Postbuckling analysis of functionally graded plates on an elastic foundation

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Abstract
First, we discuss characteristics of functionally graded materials and describe methods of their manufacturing. Then, we provide an overview of analytical and numerical methods for calculating plates, with characteristics of functionally graded materials, resting on elastic foundation. The presented numerical results have been obtained by the finite elements method, referring to post-bifurcation problems of thermally loaded plates. The first-order shear deformation theory (FSDT) has been employed. In numerical calculations we have used a new 16-node plate element, free of problems related to shear locking.

1. Introduction

1.1. Functionally graded materials

Materials can be divided into structural and functional ones. While mechanical strength is the most essential property of structural materials, functional materials change their shape and physical properties under the influence of external fields. Apart from piezoelectric and magnetostrictive materials, there are shape-memory alloys of nickel and titanium, in which the phase changes from austenite into martensite and vice versa, which plays an essential role.

Functionally graded materials (FGMs) make up a new class of composite materials known for over a decade now. FGMs are composed of ceramics and metal, where components are so distributed that external ceramic layers, exposed to high temperatures, protect internal metallic functions. The most important problem here is reduction of thermal stresses due to the presence of such elements in very high temperatures and substantial differences of component thermal expansion coefficients. When two layers of different materials are joined directly, an adverse phenomenon of interfacial stresses occurs. The application of continuous transition of one phase into another allows to reduce the effects of stresses. Graded layers are characteristic of high variability of the modulus of longitudinal elasticity and the thermal conductivity across the thickness direction of the plate. From the technological viewpoint, depositing a layer of one material on another is not an easy task.

Numerous FGM applications include medicine (orthopedic resorption controllable implants with suited mechanical properties, lens, bioceramic materials, porous ceramics, bioactive ceramics, resorptive ceramics, biostable bonding materials), machine industry (items with a varying structure, from steel in the core to carbide steel on the tool surface, assuring high ductility of the core and high hardness of the surface layer and consequent abrasion resistance, such as cutting tools, toothed wheels or turbine blades) and power industry (jet engine chambers, functionally graded components of fuel cells, thermonuclear power generation - thermal barrier protection), electronics industry (heat sensors, piezoelectric activators, optical fibers).

Presently, FGM related research and development is geared to seeking new manufacturing methods, particularly those of high repeatability and efficiency. These methods must allow the maker to control the composition, density and properties of a material along a specific direction, usually across the thickness direction of the plate. FGMs must have homogeneous compaction required for multi-component materials, repeatability and strength. Manufacturing techniques must guarantee controlled changes in composition and density, so that the product will have a specific structure and properties along a given direction. A graded controlled transition between properties of the components may allow to adjust the product to expected working conditions.

FGM manufacturing makes use of a variety of methods: (i) deposition of layers by plasma spraying or lamination of ceramic sheets, (ii) those using segregation phenomena that occur in an
initially homogeneous system due to applied external forces (sedimentation deposition, electrophoretic deposition), (iii) infiltration (of a porous pre-form by liquid metal or its gaseous phase).

Plasma spraying is a method where particles of metal powder are directed into a stream of plasma (ionized gas-like matter with electrically charged particles). Then partly or fully melted metal is deposited on a substrate at high speed.

Another FGM production method is a powder laser sintering technique combined with a system of powder dosage, in which a mixture of powders is precisely supplied by a feeding system to the place of deposition, where it is processed by a laser beam. Powders with higher melting point are deposited on materials with lower melting point.

Electrophoretic deposition uses the movement of charged particles of the dispersed phase of a colloid suspension (usually two-phase non-homogeneous mixture, with one phase dissolved in the other), affected by the electric field. In case of multi-component suspensions, each component has different mobility, hence in the course of the process, the concentration of higher mobility particles drops, while the concentration of the other components in the suspension rises.

Sedimentation is another technique of forming FGMs. In the process of sedimentation a suspension of a solid body in a liquid falls due to gravitation or inertia forces. Particles which make up a mixture move at various speeds. Suspensions with a density greater than that of the liquid are subject to sedimentation. Therefore, the process leads to a separation of inhomogeneous substances, which get separated depending on their density, shape and size of grains/particles. The parameters controlling the process include those of the powder (density, grain shape and size), and properties of the liquid (density, viscosity, wettability of powders).

Infiltration makes use of materials with components having different melting points. In the process, a preform made of a material with higher melting point and a porosity gradient, is infiltrated by another, melted component. This technique can be used to fabricate metallic-ceramic, glass–ceramic, or polymer-ceramic materials, where a preform is ceramic.

Another method of producing FGM is pressing of subsequently poured/deposited powders. Hot pressing or cold pressing combined with free sintering are two other techniques for manufacturing FGMs. Basic advantages of the process are high repeatability and speed.

Finally, the technique of ceramic sheet lamination should be mentioned. Ceramic FGMs are made by sintering thin ceramic sheets fabricated in the process of roll pressing or tape casting (casting of a synthetic material or ceramic slip on a moving steel substrate).

1.2. Calculation methods for functionally graded material plates on an elastic foundation – overview

There are many publications on static bending, free vibration and stability problems referring to various methods of calculating plate and shell elements on an elastic foundation [1–9]. Bending of plates resting on a two-parameter elastic foundation under conditions of varying temperature and humidity is discussed by Zenkour [10], Zenkour and Sobhy [11] and Zenkour and Radwan [12] define a general formulation in which components of the plate strain include higher order constituents or sinusoidal terms. Such formulation does not require a corrective coefficient as is the case in the first-order shear deformation theory. Those authors presented analytical solutions of functionally graded material plates (metallic-ceramic) with properties varying across the plate thickness. In the work [13] Zenkour and Sobhy deal with problems of the dynamics of a plate. They have carried out the deflections and stresses in functionally graded plates resting on two-parameter Pasternak foundation subjected to time harmonic thermal load. Akavci in [14] has employed the theory of shear strain of higher order (hyperbolic distribution) for a stability analysis of FGM plates resting on a two-parameter elastic foundation. The author assumes that all material constants are non-linear temperature functions. Recently, Swaminathan, Naveenkumar, Zenkour and Carrera [15] have presented comprehensively analytical and numerical solutions for a number of examples concerning the statics, dynamics and stability of functionally graded plates.

The behavior of thick plates depends on the coefficient defining a plate thickness-to-length ratio. If the coefficient is very small, the phenomenon of lock shearing may occur. To eliminate this adverse phenomenon Özdemir [16] has introduced a new type of plate finite element, free from such kind of parasitic behavior. The results given by the author suggest that the element is suitable for both static and dynamic calculations of plates on Winkler foundation.

This article focuses on the stability of functionally graded material plates resting on a two-parameter elastic foundation. In numerical calculations we use a new 16-node plate element free of lock shearing problems [17].

2. Modeling of mechanical properties

2.1. Material constants

It has been assumed that material coefficients change exponentially across the plate volume, as per Eq. (2.1). Gradation is modeled by an appropriate choice of the exponent n [18]

$$V_c = \frac{1}{2} + \frac{t_0}{t} \left( n \right)^n \left( n \geq 0 \right).$$

where $V_c$ denotes a volumetric fraction of the ceramic layer in the overall structure of FGM, $t$ denotes the plate thickness, $V_m = 1 - V_c$ denotes a volumetric fraction of the metallic layer in the overall structure.

For sandwich plates the following relation is used, determining the volumetric fraction of the metallic layer (for positive values of coordinate $z$) in the external layer

$$V_m = \frac{z - t_0}{t_1 - t_0} \left( n \right)^n.$$  

The denotations $t_0$ and $t_1$ are shown in Fig. 2.1.

If we assume that Young’s modulus $E_F$ and thermal expansion coefficient $\alpha_F$ of the material are combinations of material constants and volumetric fractions of both components in the FGM, then we have

$$E_F = E_m V_m + E_c V_c, \quad \alpha_F = \alpha_m V_m + \alpha_c V_c.$$  

Fig. 2.1. A functionally graded material plate.
where $E_m$ and $E_c$ are moduli of elasticity of metal and ceramic, respectively. An assumption that $n = 0$ means that the entire structure is made of a uniform material. As FGMs are commonly used in high-temperature conditions, under which essential changes in material properties can be expected, the impact of temperature in modeling FGM properties has to be reflected. The dependence of material properties on temperature is strongly non-linear, approximately described by these equations:

$$E_m = E_{\text{ext}} \left[ 1 + \frac{E_{\text{m,-1}}}{T_0 + \Delta T} + \frac{E_{\text{m,0}}}{(T_0 + \Delta T)^2} + \frac{E_{\text{m,1}}}{(T_0 + \Delta T)^3} \right]$$

$$\nu_{xy} = \nu_{\text{ext}} \left[ 1 + \frac{\nu_{xy,-1}}{T_0 + \Delta T} + \nu_{xy,0} (T_0 + \Delta T)^2 + \nu_{xy,1} (T_0 + \Delta T)^3 \right]$$

$$E_c = E_{\text{cl}} \left[ 1 + \frac{E_{\text{c,-1}}}{T_0 + \Delta T} + E_{\text{c,1}} (T_0 + \Delta T)^2 + E_{\text{c,2}} (T_0 + \Delta T)^3 \right]$$

$$\nu_{xy} = \nu_{\text{cl}} \left[ 1 + \frac{\nu_{xy,-1}}{T_0 + \Delta T} + \nu_{xy,0} (T_0 + \Delta T)^2 + \nu_{xy,1} (T_0 + \Delta T)^3 \right].$$

Further considerations assume that Poisson’s coefficient is not temperature dependent.

Terms of the constitutive matrix of an FGM plate model are these

$$C_{11} = C_{22} = E_f (z, T) \frac{1}{1 - \nu_f^2}; \quad C_{12} = C_{21} = \nu_f E_f (z, T) \frac{1}{1 - \nu_f^2}; \quad C_{33} = \frac{E_f (z, T)}{2(1 + \nu_f)};$$

$$C_{44} = C_{55} = \frac{C_{11}}{T}.$$

(2.4)

where $\beta = 6/5$ is a shear correction factor, necessary when the first-order shear deformation theory is used.

3. Mathematical formulation of the problem

3.1. The critical load

The critical load of a plate will be calculated using the principle of virtual work

$$\delta W_{\text{int}} = \delta W_{\text{ext}},$$

where, if no external load exists, $\delta W_{\text{ext}} = 0$. Virtual work of internal forces for a plate on a two-parameter elastic foundation, according to Reissner–Mindlin plate theory, is

$$\delta W_{\text{int}} = \int_{V_y} \delta \varepsilon \epsilon dV + \int_{V_y} \left[ k_0 \omega \delta \omega + k_1 (w_x \delta w_x + w_y \delta w_y) \right] dA$$

(3.1)

where the components of the second Piola–Kirchhoff stress tensor, written in the vector form are

$$\{S\} = \{S_x \ S_y \ S_{xy} \ S_{xz} \ S_{yz}\}$$

$$S_i = C_k E_j,$$

(3.2)

where $[C]$ is a material coefficients matrix.

If we consider large displacements, plate strains (components of the Green–Lagrange strain) are as follows:

$$E_{x\zeta} = z \theta_{x\zeta} + \frac{1}{2} w_x^2 + \alpha_y \Delta T; \quad E_{\zeta \zeta} = -z \theta_{x\zeta} + \frac{1}{2} w_x^2 + \alpha_y \Delta T$$

(3.3)

$$E_{xy} = z \theta_{xy} - z \theta_{x\zeta} + w_x w_y; \quad E_{x\zeta} = \theta_{x\zeta} + w_x; \quad E_{\zeta \zeta} = -\theta_{x\zeta} + w_y,$$

or in a general form

$$E_j = B^{(i)} \delta d_i + B^{(m)} \delta d_m + \alpha_y \Delta T,$$

(3.4)

where $[B^{(i)}]$ is a strain matrix defining the linear part of the strain,

$$B_{ik} = \frac{n_{ik}}{n_{kk}} \theta_k; \quad B_{ik} = -2n_{ik} \theta_k; \quad B_{ik} = n_{ik} \theta_k - n_{ik} \theta_k,$$

(3.5)

while the non-linear part $[B_{pq}]$ is equal to

$$B_{kkm} = \frac{1}{2} n_{kk} n_{km}; \quad B_{kmm} = \frac{1}{2} n_{km} n_{mm}; \quad B_{kkm} = n_{kk} n_{km};$$

(3.6)

$$B_{kmm} = 0; \quad B_{kkm} = 0.$$

(3.7)

We calculate the virtual strains

$$\delta E_i = B^{(i)} \delta d_i + (B_{pq} + B_{pq}) d_i \delta d_i.$$  

(3.8)

Using the Eqs. (3.2)–(3.8) we can present the virtual work thus:

$$\delta W_{\text{int}} = \int_{V_y} \left[ k_0 \omega \delta \omega + k_1 (w_x \delta w_x + w_y \delta w_y) \right] dV$$

(3.9)

where $[N_i]$ is a matrix of the shape function.

After transformations, neglecting the second and third order terms with respect to displacements and assuming the coordinate system in such a way that $-\alpha_y \Delta T f_y C_i B_{0i} dV = 0$, we get

$$\left[ K_{pk}^{(p)} + K_{pk}^{(f)} + K_{pk}^{(t)} \right] d_i,$$

(3.10)

where $[K_{pk}^{(p)}] = f_{ii} C_k B_{pk}^{(i)} dV$ is the stiffness matrix of the plate,

$$K_{pk}^{(f)} = f_{jk} [k_0 n_{jk} n_{jp} + k_1 (n_{jk} n_{jp} + n_{jk} n_{jp})] dA$$

denotes the stiffness matrix of the elastic foundation, while

$$K_{pk}^{(t)} = -\Delta T f_y \alpha_y C_i B_{0i} dV$$

is a matrix similar to the geometrical matrix for an analysis of buckling under compressive load.

Equation (3.10) formulates an eigenvalue problem, in which temperature increment $\Delta T$ is an eigenvalue, and nodal displacements determining the form of loss of stability make up an eigenvector. The eigenvalue problem has been solved using a subspace iteration algorithm, given by Bathe [19].

3.2. Non-linear equilibrium equations

Equilibrium equations in the non-linear range will also be derived by using the principle of virtual work. To solve the problem, we will apply the increment-iteration method, therefore the principle of virtual work has this form

$$\delta W_{\text{int}}^{(i)} = \delta W_{\text{ext}}^{(i)}$$

(3.11)

where $\delta W_{\text{int}}^{(i)}$ is virtual work of internal forces for an increment step $i + \Delta t$ and iteration $i + 1$, comprising the work of internal forces acting on the plate, and the work of elastic foundation:

$$\delta W_{\text{int}}^{(i)} = \int_{V_y} \left( \delta S_i \epsilon_i^{(i+1)} + \delta S_i \epsilon_i^{(i+1)} \right) dV$$

$$+ \int_{V_y} \left[ k_0 \omega \delta w_x^{(i+1)} + k_1 (w_x \delta w_x^{(i+1)} + w_y \delta w_y^{(i+1)}) \right] dA$$

$$+ \frac{1}{2} \left( \delta \Delta T \right) \frac{1}{2} \left( \delta \Delta T \right) + \frac{1}{2} \left( \delta \Delta T \right) \frac{1}{2} \left( \delta \Delta T \right)$$

(3.12)

where

$$\Delta S_i^{(i)} = \frac{1}{2} \left( \Delta T \right)^2 \frac{1}{2} \left( \Delta T \right)^2$$

(3.13)

a $\delta W_{\text{int}}^{(i)}$ is virtual work from external load. As the plate is exclusively subject to thermal load, we get

$$\delta W_{\text{int}}^{(i)} = 0.$$

(3.14)
Increments of Green–Lagrange strains, assuming only large transverse displacements and accounting for thermal loads, are as follows:

\[
\begin{align*}
\Delta E_{x}^{t+1, i+1} & = \frac{B_{i+1}}{C_{0}} \Delta \theta_{t} + \frac{1}{2} \Delta w_{x}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{x}^{t+1, i+1}, \\
\Delta E_{y}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + \frac{1}{2} \Delta w_{y}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{y}^{t+1, i+1}, \\
\Delta E_{xy}^{t+1, i+1} & = z_{i+1, i} \Delta \theta_{t} + \frac{1}{2} \Delta w_{xy}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{xy}^{t+1, i+1}, \\
\Delta E_{z}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + \frac{1}{2} \Delta w_{z}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{z}^{t+1, i+1}, \\
\Delta E_{zz}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + \frac{1}{2} \Delta w_{z}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{z}^{t+1, i+1}, \\
\Delta E_{zz}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + \frac{1}{2} \Delta w_{z}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{z}^{t+1, i+1},
\end{align*}
\]

(3.15)
or, in a general form

\[
\begin{align*}
\Delta E_{x}^{t+1, i+1} & = \frac{B_{i+1}}{C_{0}} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1}, \\
\Delta E_{y}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1}, \\
\Delta E_{xy}^{t+1, i+1} & = z_{i+1, i} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1}, \\
\Delta E_{z}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1}, \\
\Delta E_{zz}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1},
\end{align*}
\]

(3.16)

where

\[
\begin{align*}
\Delta E_{x}^{t+1, i+1} & = \frac{B_{i+1}}{C_{0}} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1}, \\
\Delta E_{y}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1}, \\
\Delta E_{xy}^{t+1, i+1} & = z_{i+1, i} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1}, \\
\Delta E_{z}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1}, \\
\Delta E_{zz}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1},
\end{align*}
\]

(3.17)

while components \([B_{ijklm}]\) are given in Eq. (3.7).

The strain increment variation is

\[
\Delta e_{x}^{t+1, i+1} = \frac{B_{i+1}}{C_{0}} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1},
\]

(3.18)

Strain in an increment step \(t + \Delta t\) and iteration \(i + 1\) is a sum of the strain in the previous iteration and strain increment, calculated by the constitutive equation

\[
\begin{align*}
\Delta e_{x}^{t+1, i+1} & = \frac{B_{i+1}}{C_{0}} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1},
\end{align*}
\]

(3.19)

and the virtual work of internal forces

\[
\begin{align*}
\delta e_{x}^{t+1, i+1} W_{int} & = \int_{V_{p}} \left[ \Delta e_{x}^{t+1, i+1} \right] \left[ K_{p}^{(R)} \Delta e_{x}^{t+1, i+1} \right] \text{d}V + \int_{V_{p}} \left[ \Delta e_{x}^{t+1, i+1} \right] \left[ K_{p}^{(E)} \Delta e_{x}^{t+1, i+1} \right] \text{d}V,
\end{align*}
\]

(3.20)

After transformations, we obtain the following system of equations:

\[
\begin{align*}
\Delta E_{x}^{t+1, i+1} & = \frac{B_{i+1}}{C_{0}} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1}, \\
\Delta E_{y}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1}, \\
\Delta E_{xy}^{t+1, i+1} & = z_{i+1, i} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1}, \\
\Delta E_{z}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1}, \\
\Delta E_{zz}^{t+1, i+1} & = -z_{i+1, i} \Delta \theta_{t} + B_{jk} \Delta w_{jk}^{t+1, i+1} + \frac{1}{2} \Delta \theta_{t} + \frac{1}{2} \Delta w_{jk}^{t+1, i+1},
\end{align*}
\]

(3.21)

is the stiffness matrix of a plate element,

\[
K_{p}^{(R)} = \int_{\Delta x} [k_{0} N_{33} N_{33} + k_{1} (N_{33,3} N_{33,3} + N_{33,3} N_{33,3})] \text{d}A,
\]

(3.23)

is the stiffness matrix of elastic foundation,

\[
F_{p} = \int_{\Delta x} [k_{0} (N_{33,3} N_{33,3} + N_{33,3} N_{33,3})] \text{d}A,
\]

(3.24)
for aluminium $E = 70 \times 10^9 \, [N/mm^2]$, $v = 0.3$, $x = 23.0 \times 10^{-6} \, [K^{-1}]$,
for corundum $E = 380 \times 10^9 \, [N/mm^2]$, $v = 0.3$, $x = 7.40 \times 10^{-6} \, [K^{-1}]$,
using the gradation model described by Eq. (2.1) for various values of
exponent $n$ and the plate slenderness ratio $a/t = 50$ and 100. The
plate temperature increments at which the plate started buckling
for various fractions of metal and ceramic are shown in Tables 4.2
and 4.3.

Our calculations confirm the findings of Zhao and others [20],
i.e. clear decrease of the critical temperature increment for $n$ lower
than 1, then a visible stability of results for the rising exponent $n$.

Calculations illustrating the impact of elastic foundation stiffness
have also been made for the case analyzed by Zhao and others
[20] (a mixture of aluminum and corundum). Table 4.4 presents
the influence of Winkler’s coefficient (the shear coefficient $k_1 = 0$),
while Table 4.5 shows the influence of shear coefficient
on critical temperature increment.

To illustrate the influence of various factors on the state of equi-
librium, we have made a few series of parametric calculations for
an FGM plate model presented by Zhao and others [20]. The
adopted material was a mixture of aluminium and corundum.
Fig. 4.1 shows the influence of shape imperfection in the form:

$$w = w_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{a}, \quad (4.1)$$

where $w$ is displacement of the central point of a square plate with
the edge $a$, $w_0$ is maximum deflection of the plate. The calculations
have been made for exponent $n = 1.0$ describing the volume of
metallic fraction.

It should be noted in Fig. 4.1 that we receive the equilibrium
paths typical for nonlinear structural response of compressed
plates including $w_0/t = 0$ where usually a post-buckling curve rep-
resents the behavior of plates beginning in the bifurcation point. It
is caused by the fact that - due to the distribution of the thermal
expansion coefficient across the plate thickness for exponent $n$
different than 0 (Eq. (2.3)) - the loading generated by the thermal stresses results not only in a compressive force but also in a bending moment. The bending moment induces deflection of the plate; in the result the structural response follows the equilibrium path similar to these obtained assuming initial deflection.

The curves of equilibrium states to the post-bifurcation path for $W_0/t = 0$, that is a plate without initial shape imperfections, are typical of plates compressed axially and in two directions. Fig. 4.2 presents curves of equilibrium states for growing values of stiffness foundation –Winkler’s coefficient and shear coefficient is shown, respectively, in Figs. 4.3 and 4.4.

We can see that in both cases the stiffness of plate-foundation system increases for growing values of stiffness foundation parameters.

5. Summary

This paper presents a theoretical formulation and the results of calculations made for thermally loaded FGM plates, resting on an elastic foundation. The relevant analysis focused on the determination of critical temperature increment, i.e. a temperature at which the plate loses its stability, while a non-linear analysis aims at temperature increases of a plate with shape imperfections. The problems are formulated within the finite elements method using 16-node Lagrange plate element. Various models of ceramic and metal fractions distribution have been used in the calculations, taking into account changes in material properties of the two fractions along with temperature increase. We have compared our results with those available in publications, and found good conformity of results for isotropic material and thin plates. For plates with smaller ratio parameters (the thickness to the length of plate) we have observed certain differences in calculated results. Tangent stresses resulting from transverse shear to some extent affect numerical results.

Postbifurcation problems require further studies employing higher order shear deformation theories. The shear correction factor is essential in the first order theory. The use of higher order theories that do not require that factor may lead to more accurate results compared to analytical calculations.

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References